Explicit expression for the generating function counting Gessel's walks

Irina Kurkova, Kilian Raschel

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Abstract

Gessel's walks are the planar walks that move within the positive quadrant $\mathbb{Z}_+^2$ by unit steps in any of the following directions: West, North-East, East and South-West. In this paper, we find an explicit expression for the trivariate generating function counting the Gessel's walks with $k \geqslant 0$ steps, which start at $(0,0)$ and end at a given point $(i,j) \in \mathbb{Z}_+^2$.

Keywords: lattice walks, generating function, Riemann boundary value problem, conformal gluing function, Weierstrass elliptic function, uniformization, Riemann surface

AMS 2000 *Subject Classification: primary 05A15; secondary 30F10, 30D05*

1 Introduction

The enumeration of lattice walks is a classical problem in combinatorics and this article is about the special case of Gessel's walks. These are the planar walks that move within the positive quadrant $\mathbb{Z}_+^2$ by unit steps in any of the following directions: West, North-East, East and South-West. To be more precise, let the walk be of length $k \geqslant 0$ and end at $(i,j) \in \mathbb{Z}_+^2$: if $i,j > 0$ then the next step can either be at $(i-1,j)$, $(i+1,j+1)$, $(i+1,j)$ or $(i-1,j-1)$; if $i > 0$ and $j = 0$ it can be at $(i-1,0)$, $(i+1,1)$ or $(i+1,0)$; if $i = 0$ and $j \geqslant 0$ it can be at $(1,j+1)$ or $(1,j)$. This is illustrated on Figure 1 below.

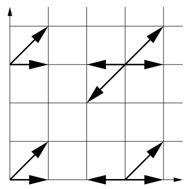

Figure 1: Steps in Gessel's walks

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For $(i,j) \in \mathbb{Z}_+^2$ and $k \geqslant 0$, let

$$q(i,j,k) = \#\big\{\text{Gessel's walks of length } k \text{ starting at } (0,0) \text{ and ending at } (i,j)\big\}.$$

These walks are named after Ira Gessel, who conjectured in 2001 that for any $k \geqslant 0$, $q(0,0,2k) = 16^k\big[(5/6)_k(1/2)_k\big]/\big[(2)_k(5/3)_k\big]$, where $(a)_k = a(a+1)\cdots(a+k-1)$. This was proven in 2009 by Kauers, Koutschan and Zeilberger [7].

Let $Q(x,y,z)$ be the trivariate generating function counting Gessel's walks:

$$Q(x,y,z) = \sum_{i,j,k\geqslant 0} q(i,j,k)x^iy^jz^k.$$

This series is entirely characterized by the generating functions $Q(x,0,z)$, $Q(0,y,z)$ and $Q(0,0,z)$—which count the walks that end on the borders of the quadrant—via the following functional equation:

$$xyz\left(\frac{1}{x}+\frac{1}{xy}+x+xy-\frac{1}{z}\right)Q(x,y,z) = zQ(x,0,z)+z(y+1)Q(0,y,z)-zQ(0,0,z)-xy. \tag{1}$$

This equation, a classical result [1, 2], is valid a priori in the domain $\{(x,y,z) : |x| \leq 1, |y| \leq 1, |z| < 1/4\}$.

In this article, with the help of complex analysis methods we obtain closed expressions for $Q(x,0,z)$, $Q(0,y,z)$ and $Q(0,0,z)$ from exploiting the functional equation (1).

By using computer calculations, Bostan and Kauers [1] showed that $Q(x,y,z)$ is algebraic and found the minimal polynomials of $Q(x,0,z)$, $Q(0,y,z)$ and $Q(0,0,z)$—by "minimal polynomial" of an algebraic function $F(u,z)$, we mean the unique monic polynomial with coefficients in $\mathbb{C}[u,z]$ dividing any polynomial which vanishes at $F(u,z)$. Van Hoeij [1] then managed to compute $Q(x,0,z)$, $Q(0,y,z)$ and $Q(0,0,z)$ as their roots.

Furthermore, Gessel's model is one of the 2^8 models of walks in the quarter plane $\mathbb{Z}_+^2$ starting from the origin and allowing steps in the interior according to a given subset of $\{-1,0,1\}^2 \setminus \{(0,0)\}$. Bousquet-Mélou and Mishna [2] provided a systematic analysis of the enumeration of these walks. After eliminating trivial models and those already solved [3] by reduction to walks in a half-plane (a simpler problem), 79 inherently different models remained. Following an idea of [4], they associated each of them with a group of birational transformations of $\mathbb{C}^2$—for details about this group see Subsection 5.1 of this article. The group is finite in 23 cases and infinite in the 56 others. Through functional equations analogous to (1), they found the generating functions for 22 of the models with an underlying finite group. The following property has been crucial to their analysis: in the (half-)orbit sum of the trivariate generating function, all terms, except for the one corresponding to the identity element of the group, have a positive part in x or in y equal to zero. This last property is not valid for Gessel's walks and consequently, Gessel's model is the only one with a finite group that is not solved in [2].

Our method of applying (1) is notably different and can be generalized to all 79 walks described above [10]. It heavily relies on the analytic approach developed by Fayolle, Iasnogorodski and Malyshev [4] and proceeds by reduction to boundary value problems

(BVPs) of Riemann-Carleman type. In the rest of the introduction we sketch this approach and explain how we adapt it to the enumeration of Gessel's walks.

The authors of [4] find explicit expressions for the generating functions of the stationary distributions $(\pi_{i,j})_{i,j\geqslant 0}$ of some ergodic random walks in the positive quadrant $\mathbb{Z}_+^2$, supposed to have four domains of spatial homogeneity: the interior $\{(i,j) : i,j > 0\}$, the real axis $\{(i,0) : i > 0\}$, the imaginary axis $\{(0,j) : j > 0\}$ and the origin $\{(0,0)\}$. In the interior, the only possible non-zero jump probabilities correspond to the eight nearest neighbors.

First, they reduce the problem to solving the following functional equation on $\{(x,y) : |x| \leq 1, |y| \leq 1\}$:

$$K(x,y)\Pi(x,y) = k(x,y)\pi(x) + \widetilde{k}(x,y)\widetilde{\pi}(y) + k_0(x,y)\pi_{0,0}, \tag{2}$$

where the polynomials $K(x,y)$, $k(x,y)$, $\widetilde{k}(x,y)$ and $k_0(x,y)$ are known, while the functions $\Pi(x,y) = \sum_{i,j\geqslant 1} \pi_{i,j}x^{i-1}y^{j-1}$, $\pi(x) = \sum_{i\geqslant 1} \pi_{i,0}x^{i-1}$ as well as $\widetilde{\pi}(y) = \sum_{j\geqslant 1} \pi_{0,j}y^{j-1}$ are unknown but holomorphic in their unit disc; the constant $\pi_{0,0}$ is unknown as well.

Second, they continue the functions $\pi(x)$ and $\widetilde{\pi}(y)$ meromorphically (with poles that can be identified) to the whole complex plane cut along some segments, see Chapter 3 of [4].

Third, they prove that both unknown meromorphic functions $\pi(x)$ and $\widetilde{\pi}(y)$ satisfy boundary value conditions of Riemann-Carleman type, see (5.1.5) on page 95 of [4]. Using information on the poles of $\pi(x)$ and $\widetilde{\pi}(y)$, they reduce the problems to finding some new holomorphic functions that are solutions to BVPs of the same type, see pages 119–124, and particularly (5.4.10). If the index χ of these BVPs is non-negative (which actually is the generic situation considered in [4]), their solutions are not unique but depend on $\chi + 1$ arbitrary constants (for the notion of the index see (5.2.7) on page 98 or (5.2.42) on page 108). Consequently, in Part 5.4, the authors of [4] reduce the BVPs above to finding holomorphic functions satisfying some new BVPs of Riemann-Carleman type with an index $\chi = -1$. Finally, these last problems are uniquely solved by converting them into BVPs of Riemann-Hilbert type, see Theorem 5.2.8 on page 108 of [4]. The functions $\pi(x)$ and $\widetilde{\pi}(y)$ can be reconstructed from these solutions; the constant $\pi_{0,0}$ is computed from the fact that $\sum_{i,j\geqslant 0} \pi_{i,j} = 1$.

Compared to (2), our equation (1) seems somewhat more difficult to analyse, since it involves an additional parameter z. On the other hand, the unknowns $zQ(x,0,z)$, $z(y+1)Q(0,y,z)$ and $zQ(0,0,z)$ of (1) have constant coefficients, unlike $\pi(x)$, $\widetilde{\pi}(y)$ and $\pi_{0,0}$ in (2). This fact implies (see Subsection 6) that $zQ(x,0,z)$ and $z(y+1)Q(0,y,z)$ can be continued to whole cut complex planes as holomorphic and not only meromorphic functions. It also entails that $zQ(x,0,z)$ and $z(y+1)Q(0,y,z)$ satisfy BVPs of Riemann-Carleman type with an index $\chi = 0$, whose solutions are unique, up to additive constants. Then, unlike [4], we don't transform these problems anymore but we solve them directly. Their solutions uniquely determine the functions $zQ(x,0,z) - zQ(0,0,z)$ and $z(y+1)Q(0,y,z) - zQ(0,0,z)$. The quantity $Q(0,0,z)$ is then found easily, e.g. from (1) by making the substitution $(x,y,z) = (0,-1,z)$—which is such that the left-hand side of (1) vanishes—see also Remark 8. Finally, $Q(x,y,z)$ is determined via (1).

2 Reduction to boundary value problems of Riemann-Carleman type

Assumption. In the sequel, we will suppose that z is fixed in $]0,1/4[$.

Before we state our main results, we must have a closer look at the kernel $L(x,y,z) = xyz\big[1/x+1/(xy)+x+xy-1/z\big]$ that appears in (1) and introduce some notations. The polynomial $L(x,y,z)$ can be written as

$$L(x,y,z) = \widetilde{a}(y,z)x^2 + \widetilde{b}(y,z)x + \widetilde{c}(y,z) = a(x,z)y^2 + b(x,z)y + c(x,z),$$

where $\widetilde{a}(y,z) = zy(y+1)$, $\widetilde{b}(y,z) = -y$, $\widetilde{c}(y,z) = z(y+1)$ and $a(x,z) = zx^2$, $b(x,z) = zx^2 - x + z$, $c(x,z) = z$. Define also

$$\widetilde{d}(y,z) = \widetilde{b}(y,z)^2 - 4\widetilde{a}(y,z)\widetilde{c}(y,z), \qquad d(x,z) = b(x,z)^2 - 4a(x,z)c(x,z).$$

We have $L(x,y,z) = 0$ if and only if $[\widetilde{b}(y,z) + 2\widetilde{a}(y,z)x]^2 = \widetilde{d}(y,z)$ or equivalently $[b(x,z) + 2a(x,z)y]^2 = d(x,z)$. In particular, the algebraic functions $X(y,z)$ and $Y(x,z)$ defined by $L(X(y,z),y,z) = 0$ and $L(x,Y(x,z),z) = 0$ respectively have two branches, namely

$$X_0(y,z) = [-\widetilde{b}(y,z) + \widetilde{d}(y,z)^{1/2}]/[2\widetilde{a}(y,z)], \quad X_1(y,z) = [-\widetilde{b}(y,z) - \widetilde{d}(y,z)^{1/2}]/[2\widetilde{a}(y,z)],$$
$$Y_0(x,z) = [-b(x,z) + d(x,z)^{1/2}]/[2a(x,z)], \quad Y_1(x,z) = [-b(x,z) - d(x,z)^{1/2}]/[2a(x,z)].$$

For any $z \in]0,1/4[$, the polynomial $\widetilde{d}$ has one root equal to zero, say $y_1(z) = 0$, as well as two real positive roots, that we denote by $y_2(z) = [1-8z^2-(1-16z^2)^{1/2}]/[8z^2]$ and $y_3(z) = [1-8z^2+(1-16z^2)^{1/2}]/[8z^2]$. We have $0 < y_2(z) < 1 < y_3(z)$. We also note $y_4(z) = \infty$. The points $y_k(z)$, $k \in \{1,\ldots,4\}$ are the four branch points of the algebraic function $X(y,z)$.

Likewise, for all $z \in]0,1/4[$, d has four real positive roots, that we denote by $x_1(z) = [1+2z-(1+4z)^{1/2}]/[2z]$, $x_2(z) = [1-2z-(1-4z)^{1/2}]/[2z]$, $x_3(z) = [1-2z+(1-4z)^{1/2}]/[2z]$ and $x_4(z) = [1+2z+(1+4z)^{1/2}]/[2z]$. We have $0 < x_1(z) < x_2(z) < 1 < x_3(z) < x_4(z)$. The points $x_k(z)$, $k \in \{1,\ldots,4\}$ are the four branch points of the algebraic function $Y(x,z)$.

We now present some properties of the two branches of both $X(y,z)$ and $Y(x,z)$.

Lemma 1. *The following properties hold.*

(i) $X_k(y,z)$, $k \in \{0,1\}$ *are meromorphic on* $\mathbb{C} \setminus ([y_1(z),y_2(z)] \cup [y_3(z),y_4(z)])$. *On the latter domain,* X_0 *has a simple zero at* -1, *no other zero and no pole;* X_1 *has a simple pole at* -1, *no other pole and no zero. Moreover, both* X_0 *and* X_1 *become infinite at* $y_1(z) = 0$ *and zero at* $y_4(z) = \infty$.

(ii) For all $y \in \mathbb{C}$, *we have* $|X_0(y,z)| \leq |X_1(y,z)|$.

(iii) $Y_k(x,z)$, $k \in \{0,1\}$ *are meromorphic on* $\mathbb{C} \setminus ([x_1(z),x_2(z)] \cup [x_3(z),x_4(z)])$. *On the latter domain,* Y_0 *has a double zero at* ∞, *no other zero and no pole;* Y_1 *has a double pole at* 0, *no other pole and no zero.*

(iv) For all $x \in \mathbb{C}$, we have $|Y_0(x,z)| \leq |Y_1(x,z)|$.

Neither $X_k(y,z)$, $k \in \{0,1\}$ is defined for y in the branch cuts $[y_1(z), y_2(z)]$ and $[y_3(z), y_4(z)]$. However, the limits $X_k^{\pm}(y,z)$ defined by $X_k^{+}(y,z) = \lim X_k(\hat{y},z)$ as $\hat{y} \to y$ from the *upper* side of the cuts and $X_k^{-}(y,z) = \lim X_k(\hat{y},z)$ as $\hat{y} \to y$ from the *lower* side of the cuts are well defined. These two quantities are each other's complex conjugate, since for y in the branch cuts, $\widetilde{d}(y,z) < 0$. A similar remark holds for $Y_k(x,z)$, $k \in \{0,1\}$ and x in $[x_1(z), x_2(z)]$ or $[x_3(z), x_4(z)]$. Precisely, for $y \in [y_1(z), y_2(z)]$ and $x \in [x_1(z), x_2(z)]$, we have

$$X_0^{\pm}(y,z) = \frac{-\widetilde{b}(y,z) \mp \imath\big[-\widetilde{d}(y,z)\big]^{1/2}}{2\widetilde{a}(y,z)}, \qquad Y_0^{\pm}(x,z) = \frac{-b(x,z) \mp \imath\big[-d(x,z)\big]^{1/2}}{2a(x,z)}, \tag{3}$$

$X_1^{\pm}(y,z) = X_0^{\mp}(y,z)$ and $Y_1^{\pm}(x,z) = Y_0^{\mp}(x,z)$. Furthermore, the identities (3) are true for $y \in [y_3(z), y_4(z)]$ and $x \in [x_3(z), x_4(z)]$ respectively if one exchanges $X_0^{\pm}(y,z)$ in $X_0^{\mp}(y,z)$ and $Y_0^{\pm}(x,z)$ in $Y_0^{\mp}(x,z)$.

In order to state our main results, we also need the following lemma.

Lemma 2. *The curves $X([y_1(z), y_2(z)], z)$ and $Y([x_1(z), x_2(z)], z)$ satisfy the following properties.*

(i) They are symmetrical w.r.t. the real axis and not included in the unit disc.

(ii) $X([y_1(z), y_2(z)], z)$ contains ∞ and $Y([x_1(z), x_2(z)], z)$ is closed.

(iii) They split the plane $\mathbb{C}$ in two connected components. We call $\mathscr{G}X([y_1(z), y_2(z)], z)$ and $\mathscr{G}Y([x_1(z), x_2(z)], z)$ the ones of the branch points $x_1(z)$ and $y_1(z)$ respectively. They are such that $[x_1(z), x_2(z)] \subset \mathscr{G}X([y_1(z), y_2(z)], z) \subset \mathbb{C} \setminus [x_3(z), x_4(z)]$ as well as $[y_1(z), y_2(z)] \subset \mathscr{G}Y([x_1(z), x_2(z)], z) \subset \mathbb{C} \setminus [y_3(z), y_4(z)]$.

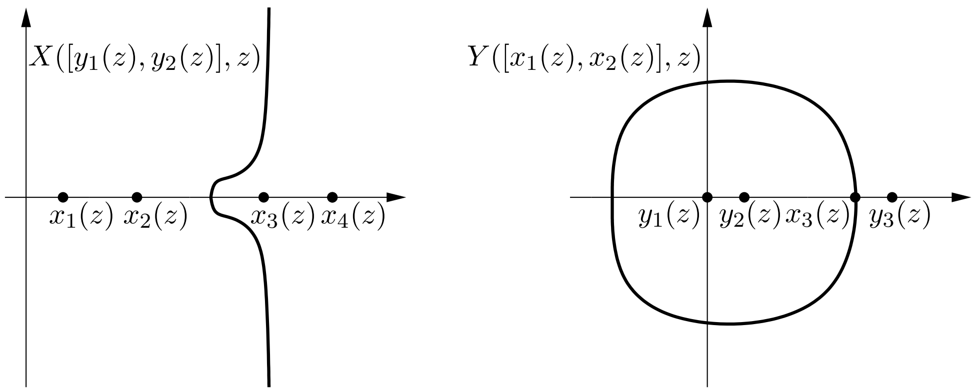

Figure 2: The curves $X([y_1(z), y_2(z)], z)$ and $Y([x_1(z), x_2(z)], z)$

The proofs of Lemmas 1 and 2 are given in Part 5.3 of [4] for $z = 1/4$; they can be plainly extended up to $z \in]0, 1/4[$.

We shall soon see that $zQ(x,0,z)$ and $z(y+1)Q(0,y,z)$ satisfy BVPs of Riemann-Carleman type. It will turn out that the underlying boundary conditions for $zQ(x,0,z)$ and $z(y+1)Q(0,y,z)$ hold *formally* on the curves $X([y_1(z),y_2(z)],z)$ and $Y([x_1(z),x_2(z)],z)$ respectively. However, these curves are *not* included in the unit disc, see Lemma 2 above, therefore the functions $zQ(x,0,z)$ and $z(y+1)Q(0,y,z)$ are a priori not defined on them. For this reason we need, first of all, to continue the generating functions up to these curves. This is why we state the following theorem; its proof is postponed to Section 6.

Theorem 3. *The functions $zQ(x,0,z)$ and $z(y+1)Q(0,y,z)$ can be continued as holomorphic functions from their unit disc up to $\mathbb{C}\setminus[x_3(z),x_4(z)]$ and $\mathbb{C}\setminus[y_3(z),y_4(z)]$ respectively. Furthermore, for any $y\in\mathbb{C}\setminus([y_1(z),y_2(z)]\cup[y_3(z),y_4(z)])$,*

$$zQ(X_0(y,z),0,z)+z(y+1)Q(0,y,z)-zQ(0,0,z)-X_0(y,z)y=0 \tag{4}$$

and for any $x\in\mathbb{C}\setminus([x_1(z),x_2(z)]\cup[x_3(z),x_4(z)])$,

$$zQ(x,0,z)+z(Y_0(x,z)+1)Q(0,Y_0(x,z),z)-zQ(0,0,z)-xY_0(x,z)=0. \tag{5}$$

It is immediate from Theorem 3 that for any $z\in]0,1/4[$, the function $Q(0,y,z)$ can be continued as a holomorphic function from the unit disc up to $\mathbb{C}\setminus[y_3(z),y_4(z)]$ as well: the point $y=-1$ cannot be a pole of $Q(0,y,z)$ since the series $\sum_{j\geqslant 0,k\geqslant 0}q(0,j,k)z^k$ converges.

Now, we derive the above-mentioned boundary conditions satisfied by the functions $zQ(x,0,z)$ and $z(y+1)Q(0,y,z)$ on the curves $X([y_1(z),y_2(z)],z)$ and $Y([x_1(z),x_2(z)],z)$ respectively.

Lemma 4. *$zQ(x,0,z)$ and $z(y+1)Q(0,y,z)$ belong to the class of the functions holomorphic in $\mathscr{G}X([y_1(z),y_2(z)],z)$ and $\mathscr{G}Y([x_1(z),x_2(z)],z)$ respectively, continuous up to the boundary of the latter sets and satisfying the boundary conditions*

$$\begin{aligned} z\big[Q\big(t,0,z\big)-Q\big(\overline{t},0,z\big)\big] &= tY_0\big(t,z\big)-\overline{t}Y_0\big(\overline{t},z\big), && \forall t\in X([y_1(z),y_2(z)],z),\\ z\big[\big(t+1\big)Q\big(0,t,z\big)-\big(\overline{t}+1\big)Q\big(0,\overline{t},z\big)\big] &= X_0\big(t,z\big)t-X_0\big(\overline{t},z\big)\overline{t}, && \forall t\in Y([x_1(z),x_2(z)],z).\end{aligned} \tag{6}$$

Proof. Due to Lemma 2 and Theorem 3, it remains to prove Equation (6) above. Let $y\in[y_1(z),y_2(z)]$ and let $\hat{y}^+$ and $\hat{y}^-$ be close to y, such that $\hat{y}^+$ is in the *upper* half-plane and $\hat{y}^-$ in the *lower* half-plane. Then we have (4) for both $\hat{y}^+$ and $\hat{y}^-$. If now $\hat{y}^+\to y$ and $\hat{y}^-\to y$, then we obtain $X_0(\hat{y}^+,z)\to X_0^+(y,z)$ and $X_0(\hat{y}^-,z)\to X_0^-(y,z)=X_1^+(y,z)$. So we have proved that for any $y\in[y_1(z),y_2(z)]$,

$$zQ(X_0^+(y,z),0,z)+z(y+1)Q(0,y,z)-zQ(0,0,z)-X_0^+(y,z)y=0, \tag{7}$$

$$zQ(X_1^+(y,z),0,z)+z(y+1)Q(0,y,z)-zQ(0,0,z)-X_1^+(y,z)y=0. \tag{8}$$

Subtracting (8) from (7) gives that for $y\in[y_1(z),y_2(z)]$,

$$z\big[Q(X_0^+(y,z),0,z)-Q(X_1^+(y,z),0,z)\big]=X_0^+(y,z)y-X_1^+(y,z)y.$$

Then, using the fact that for $k\in\{0,1\}$, $y\in[y_1(z),y_2(z)]$ and $z\in]0,1/4[$, $Y_0(X_k^\pm(y,z),z)=y$—which can be proved by elementary considerations starting from Lemma 1—we get the first part of (6). Likewise, we could prove the second part of (6). $\square$

3 Results

Problems as in Lemma 4 are usually called BVPs of Riemann-Carleman type, see e.g. Part 5.2.5 of [4]. A standard way to solve them consists in converting them into BVPs of Riemann-Hilbert type (i.e. with boundary conditions on segments) by using *conformal gluing functions* (CGFs), as in Equation (17.4') on page 130 of [5].

Definition 5. *Let $\mathscr{C} \subset \mathbb{C} \cup \{\infty\}$ be an open and simply connected set, symmetrical w.r.t. the real axis and different from $\emptyset$, $\mathbb{C}$ and $\mathbb{C} \cup \{\infty\}$. A function w is said to be a CGF for set $\mathscr{C}$ if:*

(i) w is meromorphic in $\mathscr{C}$;

(ii) w establishes a conformal mapping of $\mathscr{C}$ onto the complex plane cut along some arc;

(iii) for all t in the boundary of $\mathscr{C}$, $w(t) = w(\overline{t})$.

Let $w(t,z)$ and $\widetilde{w}(t,z)$ be CGFs for $\mathscr{G}X([y_1(z), y_2(z)], z)$ and $\mathscr{G}Y([x_1(z), x_2(z)], z)$. The existence (but *no* explicit expression) of w and $\widetilde{w}$ follows from general results on conformal gluing, see e.g. Part 17.5 in [5].

Transforming the BVPs of Riemann-Carleman type into BVPs of Riemann-Hilbert type thanks to w and $\widetilde{w}$, solving the latter and working out the solutions, we will prove the following.

Theorem 6. *For $z \in]0, 1/4[$ and $x \in \mathbb{C} \setminus [x_3(z), x_4(z)]$,*

$$z\big[Q(x,0,z) - Q(0,0,z)\big] = \\ xY_0(x,z) + \frac{1}{\pi}\int_{x_1(z)}^{x_2(z)} \frac{t\big[-d(t,z)\big]^{1/2}}{2a(t,z)}\left[\frac{\partial_t w(t,z)}{w(t,z) - w(x,z)} - \frac{\partial_t w(t,z)}{w(t,z) - w(0,z)}\right]\mathrm{d}t,$$

where w is a CGF for the set $\mathscr{G}X([y_1(z), y_2(z)], z)$.

For $z \in]0, 1/4[$ and $y \in \mathbb{C} \setminus [y_3(z), y_4(z)]$,

$$z\big[(y+1)Q(0,y,z) - Q(0,0,z)\big] = \\ X_0(y,z)y + \frac{1}{\pi}\int_{y_1(z)}^{y_2(z)} \frac{t\big[-\widetilde{d}(t,z)\big]^{1/2}}{2\widetilde{a}(t,z)}\left[\frac{\partial_t \widetilde{w}(t,z)}{\widetilde{w}(t,z) - \widetilde{w}(y,z)} - \frac{\partial_t \widetilde{w}(t,z)}{\widetilde{w}(t,z) - \widetilde{w}(0,z)}\right]\mathrm{d}t,$$

where $\widetilde{w}$ is a CGF for the set $\mathscr{G}Y([x_1(z), x_2(z)], z)$.

For $z \in]0, 1/4[$,

$$Q(0,0,z) = -\frac{1}{\pi}\int_{y_1(z)}^{y_2(z)} \frac{t\big[-\widetilde{d}(t,z)\big]^{1/2}}{2\widetilde{a}(t,z)}\left[\frac{\partial_t \widetilde{w}(t,z)}{\widetilde{w}(t,z) - \widetilde{w}(-1,z)} - \frac{\partial_t \widetilde{w}(t,z)}{\widetilde{w}(t,z) - \widetilde{w}(0,z)}\right]\mathrm{d}t,$$

where $\widetilde{w}$ is a CGF for the set $\mathscr{G}Y([x_1(z), x_2(z)], z)$.

The function $Q(x,y,z)$ has then the explicit expression obtained by using the ones of $Q(x,0,z)$, $Q(0,y,z)$ and $Q(0,0,z)$ in (1).

All functions in the integrands above are explicit, except for the CGFs w and $\widetilde{w}$. In [4], suitable CGFs are computed *implicitly* by means of the inverse of some known function, see Equations (25) and (27) in Section 5 for the details. Starting from this representation, we are able to make *explicit* these functions for the case of Gessel's walks. In order to state the result, we need to define

$$\begin{aligned} G_2(z) &= (4/27)\big(1+224z^2+256z^4\big), \\ G_3(z) &= (8/729)\big(1+16z^2\big)\big(1-24z+16z^2\big)\big(1+24z+16z^2\big). \end{aligned} \tag{9}$$

We define also $K(z)$ as the unique real positive solution to

$$K^4 - G_2(z)K^2/2 - G_3(z)K - G_2(z)^2/48 = 0. \tag{10}$$

With the notations $r_k(z) = [G_2(z) - \exp(2k\imath\pi/3)(G_2(z)^3 - 27G_3(z)^2)^{1/3}]/3$ for $k \in \{0,1,2\}$, we have $K(z) = [-r_0(z)^{1/2} + r_1(z)^{1/2} + r_2(z)^{1/2}]/2$. We finally define

$$\begin{aligned} F(t,z) &= \frac{1-24z+16z^2}{3} - \frac{4(1-4z)^2}{z}\frac{t^2}{(t-x_2(z))(t-1)^2(t-x_3(z))}, \\ \widetilde{F}(t,z) &= \frac{1-24z+16z^2}{3} + \frac{4(1-4z)^2}{z}\frac{t(t+1)^2}{[(t-x_2(z))(t-x_3(z))]^2}. \end{aligned} \tag{11}$$

Theorem 7. *A suitable CGF for the set $\mathscr{G}X([y_1(z), y_2(z)], z)$ is the unique function having a pole at $x_2(z)$ and solution to*

$$\begin{aligned} w^3 - w^2\big[F(t,z) + 2K(z)\big] + w\big[2K(z)F(t,z) + K(z)^2/3 + G_2(z)/2\big] \\ - \big[K(z)^2F(t,z) + 19G_2(z)K(z)/18 + G_3(z) - 46K(z)^3/27\big] = 0. \end{aligned} \tag{12}$$

Likewise, a suitable CGF for the set $\mathscr{G}Y([x_1(z), x_2(z)], z)$ is the unique function having a pole at $x_3(z)$ and solution to the equation obtained from (12) by replacing F by $\widetilde{F}$, see (11).

The function $Q(x,y,z)$ being found in the domain $\{|x|<1, |y|<1, z \in]0,1/4[\}$ thanks to Theorems 6 and 7, its coefficients $Q_{i,j}(z) = \sum_{k\geqslant 0} q(i,j,k)z^k$ can be obtained most easily by the use of Cauchy formulas, for any real $z \in]0,1/4[$. Since the radius of convergence of the complex series $\sum_{k\geqslant 0} q(i,j,k)z^k$ is not smaller than $1/4$, the numbers $q(i,j,k)$ of Gessel's walks can then be identified e.g. in terms of the limits of the successive derivatives as $z \to 0+$ (i.e. as $z>0$ goes to 0):

$$q(i,j,k) = \lim_{z\to 0+} \frac{1}{k!}\frac{\mathrm{d}^k Q_{i,j}(z)}{\mathrm{d}z^k}.$$

Let us now outline two facts about the expressions of $Q(x,0,z)$, $Q(0,y,z)$ and $Q(0,0,z)$ stated in Theorems 6 and 7.

Remark 8. The fact that (1) is valid at least on $\{|x|\leq 1, |y|\leq 1, |z|<1/4\}$ gives that for any triplet $(\hat{x},\hat{y},z) \in \{|x|\leq 1, |y|\leq 1, |z|<1/4\}$ such that $L(\hat{x},\hat{y},z)=0$, the right-hand side of (1) equals zero, in such a way that

$$z\big[Q(\hat{x},0,z) - Q(0,0,z)\big] + z\big[(\hat{y}+1)Q(0,\hat{y},z) - Q(0,0,z)\big] + zQ(0,0,z) - \hat{x}\hat{y} = 0. \tag{13}$$

We deduce that

$$zQ(0,0,z) = -z\big[Q(\hat{x},0,z) - Q(0,0,z)\big] - z\big[(\hat{y}+1)Q(0,\hat{y},z) - Q(0,0,z)\big] + \hat{x}\hat{y}, \quad (14)$$

where the functions in square brackets in the right-hand side of (14) are given thanks to Theorem 6.

To get the expression of $zQ(0,0,z)$ given in Theorem 6, we have chosen to substitute $(\hat{x},\hat{y},z) = (0,-1,z)$ in (14), which is suitable, since with Lemma 1 we have $X_0(-1,z) = 0$.

Moreover, a consequence of Theorem 3 is that (13) is valid not only on $\{L(x,y,z) = 0\} \cap \{|x| \leq 1, |y| \leq 1, z \in]0,1/4[\}$ but in a much larger domain of the algebraic curve $\{L(x,y,z) = 0\}$. Namely, if $(\hat{x},\hat{y},z)$ is such that $z \in]0,1/4[$ and $\hat{y} = Y_0(\hat{x},z)$ or $\hat{x} = X_0(\hat{y},z)$, then (13) is still valid. Substituting any triplet $(\hat{x},\hat{y},z)$ lying in this domain into (13) yields $zQ(0,0,z)$ as in (14).

Remark 9. In Theorem 6, $z[Q(x,0,z) - Q(0,0,z)]$ and $z[(y+1)Q(0,y,z) - Q(0,0,z)]$ are written as the sums of two functions not holomorphic but singular near $[x_1(z), x_2(z)]$ and $[y_1(z), y_2(z)]$ respectively. The sums of these two singular functions are of course holomorphic near these segments, since the latter are included in the unit disc, according to Section 2. By an application of the residue theorem as in Section 4 of [8], we could write both generating functions as functions manifestly holomorphic near these segments and having in fact their singularities near $[x_3(z), x_4(z)]$ and $[y_3(z), y_4(z)]$ respectively.

We conclude the discussion of Theorems 3, 6 and 7 with the following remark.

Remark 10. With the analytical approach proposed in this article, it would be possible, without additional difficulty, to obtain explicitly the generating function of the number of walks of length k, starting at an arbitrary initial state (i_0, j_0) and ending at (i,j). Indeed, the only difference is that the product xy in (1) would then be replaced by $x^{i_0+1}y^{j_0+1}$.

The rest of the article is organized as follows. In Section 4, we prove Theorem 6. In Section 5, we give the proof of Theorem 7. There, the general implicit representation of the CGFs inspired by [4] is given and developed for the case of Gessel's walks. The proof of Theorem 3 is postponed to the last Section 6.

4 Proof of Theorem 6

The proof is composed of three steps.

Step 1. We solve the BVPs of Riemann-Carleman type with conditions (6) by transforming them into BVPs of Riemann-Hilbert type as in Parts 5.2.3–5.2.5 of [4] or in Part 17(.5) of [5]. The only notable difference from [4] is that the index of our problems is zero; this is why the solution $zQ(x,0,z)$ is found in $\mathscr{G}X([y_1(z), y_2(z)], z)$ up to an additive function of z, as

$$zQ(x,0,z) = \frac{1}{2\pi\imath}\int_{X([y_1(z),y_2(z)],z)} tY_0(t,z)\frac{\partial_t w(t,z)}{w(t,z) - w(x,z)}\mathrm{d}t, \quad \forall x \in \mathscr{G}X([y_1(z), y_2(z)], z) \quad (15)$$

where w is the CGF used for $\mathscr{G}X([y_1(z), y_2(z)], z)$. Similarly, we could write an integral representation for $z(y+1)Q(0, y, z)$, up to some additive function of z.

Step 2. We transform these representations into the integrals on real segments written in the statement of Theorem 6. This step is inspired by [8].

Let $C(\epsilon, z)$ be any contour such that:

(i) $C(\epsilon, z)$ is connected and contains ∞;

(ii) $C(\epsilon, z) \subset (\mathscr{G}X([y_1(z), y_2(z)], z) \cup X([y_1(z), y_2(z)], z)) \setminus [x_1(z), x_2(z)]$;

(iii) $\lim_{\epsilon \to 0} C(\epsilon, z) = X([y_1(z), y_2(z)], z) \cup S(z)$, with $S(z)$ the segment $[x_1(z), X(y_2(z), z)]$ traversed from $X(y_2(z), z)$ to $x_1(z)$ along the lower edge of the slit and then back to $X(y_2(z), z)$ along the upper edge.

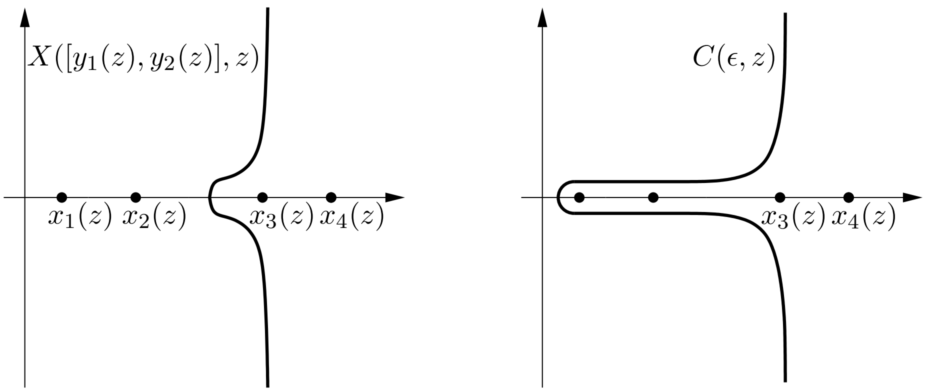

Figure 3: The curve $X([y_1(z), y_2(z)], z)$ and the new contour of integration $C(\epsilon, z)$

Let $\mathscr{G}C(\epsilon, z)$ be the connected component of $\mathbb{C} \setminus C(\epsilon, z)$ which does not contain $x_3(z)$. Now we apply the residue theorem to the integrand of (15) on the contour $C(\epsilon, z)$. Thanks to Lemma 1 and Property (ii) above, $t \mapsto tY_0(t, z)$ is holomorphic in $\mathscr{G}C(\epsilon, z)$. Likewise, by using Definition 5 and Property (ii), we reach the conclusion that $\partial_t w(t, z)/[w(t, z) - w(x, z)]$ is meromorphic on $\mathscr{G}C(\epsilon, z)$, with a single pole at $t = x$. For this reason,

$$\frac{1}{2\pi\imath} \int_{C(\epsilon, z)} tY_0(t, z) \frac{\partial_t w(t, z)}{w(t, z) - w(x, z)} \mathrm{d}t = xY_0(x, z). \tag{16}$$

Then, letting ϵ tend to 0, using Equations (15)–(16) and Property (iii) of the contour, we derive that, up to an additive function of z,

$$zQ(x, 0, z) = xY_0(x, z) - \frac{1}{2\pi\imath} \int_{S(z)} tY_0(t, z) \frac{\partial_t w(t, z)}{w(t, z) - w(x, z)} \mathrm{d}t. \tag{17}$$

Since for any $x \in \mathscr{G}X([y_1(z), y_2(z)], z)$, the integrand of (17) is, as a function of the variable t, holomorphic in $]x_2(z), X(y_2(z), z)[$, we have

$$\int_{S(z)} tY_0(t, z) \frac{\partial_t w(t, z)}{w(t, z) - w(x, z)} \mathrm{d}t = \int_{x_1(z)}^{x_2(z)} \left[tY_0^+(t, z) - tY_0^-(t, z) \right] \frac{\partial_t w(t, z)}{w(t, z) - w(x, z)} \mathrm{d}t.$$

Using (3), we then immediately obtain the expression of $z[Q(x,0,z)-Q(0,0,z)]$ stated in Theorem 6 for $x\in\mathscr{G}X([y_1(z),y_2(z)],z)$. Likewise, we could obtain the expression of $z[(y+1)Q(0,y,z)-Q(0,0,z)]$ written in Theorem 6 for $y\in\mathscr{G}Y([x_1(z),x_2(z)],z)$. The formula for $Q(0,0,z)$ has already been proved in Remark 8.

Step 3. In order to complete the proof of Theorem 6, we have to show that the integral representations of $z[Q(x,0,z)-Q(0,0,z)]$ and $z[(y+1)Q(0,y,z)-Q(0,0,z)]$ hold not only on $\mathscr{G}X([y_1(z),y_2(z)],z)$ and $\mathscr{G}Y([x_1(z),x_2(z)],z)$ but on the domains $\mathbb{C}\setminus[x_3(z),x_4(z)]$ and $\mathbb{C}\setminus[y_3(z),y_4(z)]$ respectively.

It is clear that they can be continued up to $\mathbb{C}\setminus([x_3(z),x_4(z)]\cup(w^{-1}(w([x_1(z),x_2(z)],z))\setminus[x_1(z),x_2(z)],z))$ and $\mathbb{C}\setminus([y_3(z),y_4(z)]\cup(\widetilde{w}^{-1}(\widetilde{w}([y_1(z),y_2(z)],z))\setminus[y_1(z),y_2(z)]))$ respectively. To conclude the proof of Theorem 6, it therefore suffices to show that $w^{-1}(w([x_1(z),x_2(z)],z))\setminus[x_1(z),x_2(z)]=\emptyset$ and $\widetilde{w}^{-1}(\widetilde{w}([y_1(z),y_2(z)],z))\setminus[y_1(z),y_2(z)]=\emptyset$. This is the subject of Proposition 17; it is postponed to Section 5 because all necessary facts about the functions w and $\widetilde{w}$ are proved there. ☐

5 Study of the conformal gluing functions

Notation. To be concise we drop, from now on, the dependence on z of all quantities.

The main subject of Section 5 is to prove Theorem 7. For this we shall define two functions, namely w in (25) and $\widetilde{w}$ in (27), which thanks to Part 5.5 of [4] are known to be suitable CGFs for the sets $\mathscr{G}X([y_1,y_2])$ and $\mathscr{G}Y([x_1,x_2])$ respectively. We shall then show that these functions satisfy the conclusions of Theorem 7.

5.1 Uniformization

Let us begin Section 5 with studying a *uniformization* of the algebraic curve $\mathscr{L}=\{(x,y)\in(\mathbb{C}\cup\{\infty\})^2 : L(x,y,z)=0\}$, where $L(x,y,z)=xyz\big[1/x+1/(xy)+x+xy-1/z\big]$ is the kernel appearing in (1).

Proposition 11. *For any $z\in]0,1/4[$, $\mathscr{L}$ is a Riemann surface of genus one.*

Proof. We have shown in Section 2 that $L(x,y,z)=0$ if and only if $[b(x)+2a(x)y]^2=d(x)$. The Riemann surface of the square root of a polynomial which has four distinct roots of order one having genus one (see e.g. Part 4.9 in [6], particularly pages 162–163), the genus of $\mathscr{L}$ is also one. ☐

With Proposition 11, it is immediate that $\mathscr{L}$ is isomorphic to some torus. In other words, there exists a two-dimensional lattice Ω such that $\mathscr{L}$ is isomorphic to $\mathbb{C}/\Omega$. Such a suitable lattice Ω (in fact the *only possible* lattice, up to a homothetic transformation) is found explicitly in Parts 3.1 and 3.3 of [4], namely $\omega_1\mathbb{Z}+\omega_2\mathbb{Z}$, where

$$\omega_1=\imath\int_{x_1}^{x_2}\frac{\mathrm{d}x}{[-d(x)]^{1/2}},\qquad \omega_2=\int_{x_2}^{x_3}\frac{\mathrm{d}x}{[d(x)]^{1/2}}. \tag{18}$$

We shall now give a uniformization of surface $\mathscr{L}$, i.e. we shall make explicit $x(\omega)$ and $y(\omega)$, two functions elliptic w.r.t. lattice Ω and such that $\mathscr{L} = \{(x(\omega), y(\omega)), \omega \in \mathbb{C}/\Omega\}$. By using the same arguments as in Part 3.3 of [4], we immediately see that we can take

$$x(\omega) = x_4 + \frac{d'(x_4)}{\wp(\omega) - d''(x_4)/6}, \quad y(\omega) = \frac{1}{2a(x(\omega))}\left[-b(x(\omega)) + \frac{d'(x_4)\wp'(\omega)}{2[\wp(\omega) - d''(x_4)/6]^2}\right], \tag{19}$$

where $\wp$ is the Weierstrass elliptic function with periods ω_1, ω_2.

It is well-known (see e.g. (6.7.26) on page 159 in [9]) that $\wp$ is characterized by its invariants g_2, g_3 through

$$\wp'(\omega)^2 = 4\wp(\omega)^3 - g_2\wp(\omega) - g_3. \tag{20}$$

Lemma 12. *Invariants g_2, g_3 of $\wp$ are equal to:*

$$g_2 = (4/3)\big(1 - 16z^2 + 16z^4\big), \qquad g_3 = -(8/27)\big(1 - 8z^2\big)\big(1 - 16z^2 - 8z^4\big). \tag{21}$$

Proof. We have that $4\wp(\omega)^3 - g_2\wp(\omega) - g_3 = 4[\wp(\omega) - \wp(\omega_1/2)][\wp(\omega) - \wp([\omega_1+\omega_2]/2)][\wp(\omega) - \wp(\omega_2/2)]$, see e.g. (6.7.16) on page 158 and (6.7.26) on page 159 in [9]. In particular, invariants g_2, g_3 can be expressed in terms of the values of $\wp$ at the half-periods. But it is clear by construction—and proved in Part 3.3 of [4]—that setting

$$f(t) = \frac{d''(x_4)}{6} + \frac{d'(x_4)}{t - x_4}, \tag{22}$$

we have $\wp(\omega_1/2) = f(x_3)$, $\wp([\omega_1+\omega_2]/2) = f(x_2)$ and $\wp(\omega_2/2) = f(x_1)$. Lemma 12 follows then from a direct calculation. $\square$

For an upcoming use, we would like now to know the inverse images through the uniformization of the important cycles that are the branch cuts, that is to say $x^{-1}([x_1, x_2])$, $x^{-1}([x_3, x_4])$, $y^{-1}([y_1, y_2])$ and $y^{-1}([y_3, y_4])$. In this perspective, we introduce a new period, namely

$$\omega_3 = \int_{-\infty}^{x_1} \frac{\mathrm{d}x}{[d(x)]^{1/2}}. \tag{23}$$

We will extensively use that $\omega_3 \in]0, \omega_2[$—this fact is proved in Lemma 3.3.3 on page 47 of [4].

Proposition 13. *We have $x^{-1}([x_1, x_2]) = [0, \omega_1[+ \omega_2/2$ as well as $x^{-1}([x_3, x_4]) = [0, \omega_1[$, $y^{-1}([y_1, y_2]) = [0, \omega_1[+ [\omega_2 + \omega_3]/2$ and $y^{-1}([y_3, y_4]) = [0, \omega_1[+ \omega_3/2$.*

Proposition 13 follows from repeating the arguments given in Part 3.3 of [4]. It is illustrated in Figure 5.

Let $S(x, y) = 1/x + 1/(xy) + x + xy$ be the jump generating function of Gessel's walks. Consider the two birational transformations of $(\mathbb{C} \cup \{\infty\})^2$

$$\Psi(x, y) = \left(x, \frac{1}{x^2 y}\right), \qquad \Phi(x, y) = \left(\frac{1}{xy}, y\right).$$

They satisfy $\Psi^2 = \Phi^2 = \mathrm{id}$ and $S \circ \Psi = S \circ \Phi = S$. Then, as in Part 2.4 of [4], we define the *group of the walk* as group G generated by Ψ and Φ. It is shown in [2] that for Gessel's walks, G is of order eight: in other words, $\inf\{n > 0 : (\Phi \circ \Psi)^n = \mathrm{id}\} = 4$.

If $(x, y) \in (\mathbb{C} \cup \{\infty\})^2$ is such that $L(x, y, z) = 0$ and if θ is any element of G, then clearly $L(\theta(x, y), z) = 0$. This implies that group G can also be understood as a group of automorphisms of the algebraic curve $\mathscr{L}$. It is then shown in (3.1.6) and (3.1.8) in Part 3.1 of [4] that the automorphisms Ψ and Φ of $\mathscr{L}$ become the automorphisms of $\mathbb{C}/\Omega$

$$\psi(\omega) = -\omega, \qquad \phi(\omega) = -\omega + \omega_3 \tag{24}$$

respectively. They are such that $\psi^2 = \phi^2 = \mathrm{id}$, $x \circ \psi = x$, $y \circ \psi = 1/(x^2 y)$, $x \circ \phi = 1/(xy)$ and $y \circ \phi = y$.

A crucial fact is the following.

Proposition 14. *For all $z \in]0, 1/4[$, we have $\omega_3 = 3\omega_2/4$.*

Proof. Since the group generated by Ψ and Φ is of order eight, the group generated by ψ and ϕ is also of order eight, for any $z \in]0, 1/4[$, see Section 3 of [10]. In other words, $\inf\{n > 0 : (\phi \circ \psi)^n = \mathrm{id}\} = 4$. With (24), this immediately implies that $4\omega_3$ is some point of the lattice Ω, contrary to ω_3, $2\omega_3$ and $3\omega_3$. But we already know that $\omega_3 \in]0, \omega_2[$, so that two possibilities remain: either $\omega_3 = \omega_2/4$ or $\omega_3 = 3\omega_2/4$.

In addition, essentially because the covariance of Gessel's walks is positive, we can use the same arguments as in Section 4 of [8] and this way, we conclude that ω_3 has to be larger than $\omega_2/2$, which finally yields Proposition 14. □

5.2 Global properties of CGFs

As said in Section 3, the *existence* of CGFs for the sets $\mathscr{G}X([y_1, y_2])$ and $\mathscr{G}Y([x_1, x_2])$ follows from general results on conformal gluing, see e.g. page 130 of Part 17.5 in [5]. Finding *explicit* expressions for CGFs is far more problematic: indeed, except for a few particular cases, like e.g. discs or ellipses, obtaining the expression of a CGF for a given set is, in general, quite a difficult task.

But by using the same analysis as in Part 5.5 of [4], we obtain explicitly suitable CGFs for $\mathscr{G}X([y_1, y_2])$ and $\mathscr{G}Y([x_1, x_2])$. Before writing the expression of these CGFs, we recall that $\wp$, and thus also x with (19), take each value of $\mathbb{C} \cup \{\infty\}$ twice on $[0, \omega_2[\times [0, \omega_1/\imath[$, but are one-to-one from the half-parallelogram $(]0, \omega_2/2[\times [0, \omega_1/\imath[) \cup [0, \omega_1/2] \cup ([0, \omega_1/2] + \omega_2/2)$ onto $\mathbb{C} \cup \{\infty\}$—indeed, see Corollary 3.10.8 in [6] and remember that $\wp$ is even. In particular, on the latter domain, x admits an inverse function, that we denote by x^{-1}.

Then, with Part 5.5.2.1 of [4], we state:

$$w(t) = \wp_{1,3}\big(x^{-1}(t) - [\omega_1 + \omega_2]/2\big), \tag{25}$$

where $\wp_{1,3}$ is the Weierstrass elliptic function with periods ω_1, ω_3, x^{-1} the inverse function of the first coordinate of the uniformization (19) and where $\omega_1, \omega_2, \omega_3$ are defined in (18) and (23). With (19), we note that

$$x^{-1}(t) = \wp^{-1}(f(t)), \tag{26}$$

where f is defined in (22).

In Section 4 of [8], we have studied some properties of the function w defined in (25) and we have shown that if $\omega_3 > \omega_2/2$ (which is actually the case here, see Proposition 14), then w is meromorphic on $\mathbb{C} \setminus [x_3, x_4]$ and has there a single pole, which is at x_2.

Let us now notice that

$$\widetilde{w}(t) = w(X_0(t)) \tag{27}$$

is a suitable CGF for the set $\mathscr{G}Y([x_1, x_2])$. Indeed, as we verify below, the three items of Definition 5 are satisfied.

Firstly, we immediately deduce from Lemma 16 and from the inclusion $\mathscr{G}Y([x_1, x_2]) \subset \mathbb{C}\setminus[y_3, y_4]$ stated in (iii) of Lemma 2 that $\widetilde{w}$ defined in (27) is meromorphic on $\mathscr{G}Y([x_1, x_2])$.

Secondly, using that w is a CGF for $\mathscr{G}X([y_1, y_2])$ as well as Property (i) of Lemma 15, we reach the conclusion that $\widetilde{w}$ establishes a conformal mapping of $\mathscr{G}Y([x_1, x_2])$ onto the complex plane cut along some arc.

Thirdly, once again with Property (i) of Lemma 15, we get $X_0(Y_0(x)) = X_0(Y_1(x)) = x$ for $x \in [x_1, x_2]$. As an immediate consequence, for $t \in Y([x_1, x_2])$ we have $\widetilde{w}(t) = \widetilde{w}(\overline{t})$.

Lemma 15. *The two following properties hold.*

(i) $X_0 : \mathscr{G}Y([x_1, x_2]) \setminus [y_1, y_2] \to \mathscr{G}X([y_1, y_2]) \setminus [x_1, x_2]$ *and* $Y_0 : \mathscr{G}X([y_1, y_2]) \setminus [x_1, x_2] \to \mathscr{G}Y([x_1, x_2]) \setminus [y_1, y_2]$ *are conformal and inverse to one another.*

(ii) $X_0(\mathbb{C}) \subset \mathbb{C} \setminus [x_3, x_4]$ *and* $Y_0(\mathbb{C}) \subset \mathbb{C} \setminus [y_3, y_4]$.

The proof of Lemma 15 is done in Part 5.3 of [4] for $z = 1/4$; it can be generalized directly up to $z \in]0, 1/4[$.

Lemma 16. *The function* $\widetilde{w}$ *defined in (27) is meromorphic on* $\mathbb{C} \setminus [y_3, y_4]$ *and has there a single pole, which is at* x_3.

Proof. X_0 being meromorphic on $\mathbb{C} \setminus ([y_1, y_2] \cup [y_3, y_4])$ and w on $\mathbb{C} \setminus [x_3, x_4]$, the function $\widetilde{w}$ defined in (27) is a priori meromorphic on $\mathbb{C} \setminus ([y_1, y_2] \cup [y_3, y_4] \cup X_0^{-1}([x_3, x_4]))$. But on the one hand, with (ii) of Lemma 15, $X_0^{-1}([x_3, x_4]) = \emptyset$. And on the other hand, thanks to the gluing property of the CGF w, $\widetilde{w}$ satisfies $\widetilde{w}^+(t) = \widetilde{w}^-(t)$ for $t \in [y_1, y_2]$, i.e. the limits of $\widetilde{w}(u)$ when $u \to t \in [y_1, y_2]$ from the upper and lower sides of the cut are equal. $\widetilde{w}$ is thus also meromorphic in a neighborhood of $[y_1, y_2]$, see e.g. Parts 5.2.3 and 5.2.4 of [4]. Finally, $\widetilde{w}$ is meromorphic on $\mathbb{C} \setminus [y_3, y_4]$.

Moreover, with (27) and since w has on $\mathbb{C} \setminus [x_3, x_4]$ only one pole, which happens to be at x_2, the only poles of $\widetilde{w}$ are at the points t where $X_0(t) = x_2$. It is then easy to verify, by a direct calculation, that x_3 is the only solution to the latter equation. $\square$

We prove now the following proposition, which completes the proof of Theorem 6.

Proposition 17. *We have* $w^{-1}(w([x_1, x_2])) \setminus [x_1, x_2] = \emptyset$ *and* $\widetilde{w}^{-1}(\widetilde{w}([y_1, y_2])) \setminus [y_1, y_2] = \emptyset$.

Proof. In order to prove the first identity, it is enough to show that for any fixed $u \in [x_1, x_2]$, the only solution in t of $w(t) = w(u)$ is $t = u$.

If $u \in [x_1, x_2]$, then $w(u) \in \wp_{1,3}([-\omega_1/2, \omega_1/2])$, see Proposition 13 and Equation (25). Thus once again with (25), the equation $w(t) = w(u)$ can be interpreted as

$$\wp_{1,3}(\omega) = \wp_{1,3}(\Upsilon), \tag{28}$$

where $\omega = x^{-1}(t) - [\omega_1 + \omega_2]/2$ and $\Upsilon \in [-\omega_1/2, \omega_1/2]$.

A priori, Equation (28) admits the solutions $\omega = \pm\Upsilon + k_1\omega_1 + k_3\omega_3$, with $k_1, k_3 \in \mathbb{Z}$—see Corollary 3.10.8 in [6], then remember that $\wp_{1,3}$ is even and periodic w.r.t. ω_1, ω_3. But in our case, ω belongs to a restricted region, namely $]-\omega_3, 0] \times [-\omega_1/(2\imath), \omega_1/(2\imath)]$. Indeed, as already noted in this section, we have $x^{-1}(\mathbb{C} \cup \{\infty\}) \subset [0, \omega_2/2] \times [0, \omega_1/\imath]$, so that thanks to Proposition 14, we have $x^{-1}(\mathbb{C} \cup \{\infty\}) - [\omega_1 + \omega_2]/2 \subset] - \omega_3, 0] \times [-\omega_1/(2\imath), \omega_1/(2\imath)]$. In the latter restricted domain, the only solutions to (28) are $\omega = \pm\Upsilon$. In particular, we get $x^{-1}(t) - [\omega_1 + \omega_2]/2 = \pm(x^{-1}(u) - [\omega_1 + \omega_2]/2)$, which yields $x^{-1}(t) = (1 \mp 1)[\omega_1 + \omega_2]/2 \pm x^{-1}(u)$.

Then, taking the image of the previous equality trough $\wp$ and using Equation (26), we obtain $f(t) = \wp((1 \mp 1)[\omega_1 + \omega_2]/2 \pm x^{-1}(u))$. Since $\wp$ is periodic w.r.t. ω_1, ω_2 and even, we conclude that $f(t) = f(u)$ and finally that $t = u$, since f is one-to-one, see (22).

The proof of the identity $w^{-1}(w([x_1, x_2])) \setminus [x_1, x_2] = \emptyset$ is concluded. Similar reasoning and the use of Equation (27) yield that $\widetilde{w}^{-1}(\widetilde{w}([y_1, y_2])) \setminus [y_1, y_2] = \emptyset$. □

5.3 Proof of Theorem 7

Our aim is to show that the function defined in (25) is the only function having a pole at x_2 and solution to (12).

Denote $\omega_4 = \omega_2/4$ and let $\wp_{1,4}$ be the Weierstrass elliptic function with periods ω_1, ω_4. We recall that $\wp$ and $\wp_{1,3}$ are the Weierstrass elliptic functions with respective periods ω_1, ω_2 and ω_1, ω_3, where $\omega_3 = 3\omega_2/4$ thanks to Proposition 14.

To begin with, let us prove the following lemma.

Lemma 18. *Let $\breve{\wp}$ be the Weierstrass elliptic function with periods noted $\hat{\omega}, \breve{\omega}$ and let n be a positive integer. Then the Weierstrass elliptic function with periods $\hat{\omega}, \breve{\omega}/n$ can be written in terms of $\breve{\wp}$ as follows:*

$$\breve{\wp}(\omega) + \sum_{k=1}^{n-1} \left[\breve{\wp}(\omega + k\breve{\omega}/n) - \breve{\wp}(k\breve{\omega}/n)\right]. \tag{29}$$

Proof. It is easy to verify that both the Weierstrass elliptic function having for periods $\hat{\omega}, \breve{\omega}/n$ and the function defined by (29) satisfy the three properties hereafter: they are elliptic with periods $\hat{\omega}, \breve{\omega}/n$; they have only one pole in the fundamental parallelogram $\hat{\omega}[0, 1[+(\breve{\omega}/n)[0, 1[$, this pole is at 0 and is of order two; they admit an expansion at $\omega = 0$ equal to $1/\omega^2 + O(\omega^2)$. Therefore, they must coincide, see e.g. Part 8.10 on pages 227–230 in [9]. □

Now we notice that by applying the following addition formula (see (6.8.10) on page 162 in [9])

$$\wp(\omega + \widetilde{\omega}) = -\wp(\omega) - \wp(\widetilde{\omega}) + \frac{1}{4}\left[\frac{\wp'(\omega) - \wp'(\widetilde{\omega})}{\wp(\omega) - \wp(\widetilde{\omega})}\right]^2, \qquad \forall \omega, \widetilde{\omega}, \tag{30}$$

to the Weierstrass elliptic function $\breve{\wp}$ in (29) and by then using the identity (20), we can express the Weierstrass elliptic function with periods $\hat{\omega}, \check{\omega}/n$ as a rational function of the Weierstrass elliptic function $\breve{\wp}$ with periods $\hat{\omega}, \check{\omega}$.

We shall apply this procedure in the proof of Lemmas 19 and 20.

Lemma 19. *We have*

$$\wp_{1,4}(\omega) = -2\wp(\omega) + \frac{\wp'(\omega)^2 + \wp'(\omega_2/4)^2}{2[\wp(\omega) - \wp(\omega_2/4)]^2} + \frac{\wp'(\omega)^2}{4[\wp(\omega) - \wp(\omega_2/2)]^2} - \wp(\omega_2/2) - 2\wp(\omega_2/4), \forall \omega \tag{31}$$

where $\wp(\omega_2/2) = f(x_1)$, $\wp(\omega_2/4) = (1+4z^2)/3$, $\wp'(\omega_2/4) = -8z^2$ *and where* $\wp'(\omega)$ *can be expressed in terms of* $\wp(\omega)$ *and* z *with Equations (20) and (21).*

Furthermore,

$$\wp_{1,4}\big(\wp^{-1}(f(t)) - [\omega_1 + \omega_2]/2\big) = F(t), \qquad \forall t \in \mathbb{C}, \tag{32}$$

where F *is defined in (11) and* f *in (22).*

Proof. We have $\omega_4 = \omega_2/4$ by definition of ω_4. Then, with (29), we can write

$$\wp_{1,4}(\omega) = \wp(\omega) + \wp(\omega+\omega_2/2) + \wp(\omega+\omega_2/4) + \wp(\omega+3\omega_2/4) - \wp(\omega_2/2) - \wp(\omega_2/4) - \wp(3\omega_2/4).$$

Using then addition formula (30) for $\wp$ as well as the three equalities $\wp(\omega_2/4) = \wp(3\omega_2/4)$, $\wp'(\omega_2/4) = -\wp'(3\omega_2/4)$ and $\wp'(\omega_2/2) = 0$—obtained from the facts that $\wp(\omega_2/2+\omega)$ is even and $\wp'(\omega_2/2+\omega)$ is odd, see (6.8.12) on page 162 in [9]—we get (31).

Using the formula below (see e.g. Exercise 8 on page 182 in [9])

$$\wp(\omega_2/4) = \wp(\omega_2/2) + \big[(\wp(\omega_2/2) - \wp(\omega_1/2))(\wp(\omega_2/2) - \wp([\omega_1+\omega_2]/2))\big]^{1/2} \tag{33}$$

as well as $\wp(\omega_1/2) = f(x_3)$, $\wp([\omega_1+\omega_2]/2) = f(x_2)$ and $\wp(\omega_2/2) = f(x_1)$, see the proof of Lemma 12, we immediately find $\wp(\omega_2/4) = (1+4z^2)/3$. With (20) and (21), we derive $\wp'(\omega_2/4)^2 = 64z^4$. Since $\wp$ is decreasing on $]0, \omega_2/2[$, see e.g. Part 6.11 on pages 166–167 in [9], we have $\wp'(\omega_2/4) < 0$ and therefore $\wp'(\omega_2/4) = -8z^2$.

Formula (31) with the known values of $\wp(\omega_2/2)$, $\wp(\omega_2/4)$, $\wp'(\omega_2/4)$ as well as with $\wp'(\omega)$ expressed in terms of $\wp(\omega)$ and z thanks to (20) and (21) gives a representation of $\wp_{1,4}(\omega)$ as a rational function of $\wp(\omega)$.

Evaluating this representation at $\omega = \wp^{-1}(f(t)) - [\omega_1+\omega_2]/2$, once again using (30) for the function $\wp$ together with (20) and (21) for the derivatives as well as the explicit value of $\wp([\omega_1+\omega_2]/2)$ given above, we get (32), after a substantial but elementary calculation. $\square$

Lemma 20. *We have*

$$\wp_{1,4}(\omega) = -\wp_{1,3}(\omega) + \frac{\wp_{1,3}'(\omega)^2 + \wp_{1,3}'(\omega_3/3)^2}{2[\wp_{1,3}(\omega) - \wp_{1,3}(\omega_3/3)]^2} - 4\wp_{1,3}(\omega_3/3), \qquad \forall \omega. \tag{34}$$

Proof. Formulas (29) and (30) combined with the fact that $\omega_4 = \omega_2/4 = \omega_3/3$, see Proposition 14, easily lead to (34). $\square$

Equality (20) for $\wp_{1,3}$, written as

$$\wp_{1,3}'(\omega)^2 = 4\wp_{1,3}(\omega)^3 - g_{2,1,3}\wp_{1,3}(\omega) - g_{3,1,3}, \tag{35}$$

allows us to express $\wp_{1,3}'(\omega)^2$ in terms of $\wp_{1,3}(\omega)$ and invariants $g_{2,1,3}, g_{3,1,3}$ associated with $\wp_{1,3}$. The next lemma gives their expression in terms of z.

Lemma 21. *Invariants $g_{2,1,3}, g_{3,1,3}$ of $\wp_{1,3}$ have the following explicit expressions:*

$$g_{2,1,3} = 40\wp_{1,3}(\omega_3/3)^2/3 - G_2, \qquad g_{3,1,3} = -280\wp_{1,3}(\omega_3/3)^3/27 + 14\wp_{1,3}(\omega_3/3)G_2/9 + G_3,$$

where G_2, G_3 are defined in (9).

Proof. The proof consists in expanding $\wp_{1,4}(\omega)$ at $\omega = 0$ in two different ways.

Firstly, we use equality (31) with $\wp'(\omega)$ expressed in terms of $\wp(\omega)$ thanks to (20), with g_2, g_3 obtained in (21) as well as with $\wp(\omega_2/2)$, $\wp(\omega_2/4)$ and $\wp'(\omega_2/4)$ found in Lemma 19. Expanding this identity in a neighborhood of $\omega = 0$, we obtain:

$$\wp_{1,4}(\omega) = \frac{1}{\omega^2} + \big[9G_2/20\big]\omega^2 - \big[27G_3/28\big]\omega^4 + O(\omega^6). \tag{36}$$

Secondly, we expand $\wp_{1,4}(\omega)$ at $\omega = 0$ using Equation (34) with $\wp_{1,3}'(\omega)$ and $\wp_{1,3}'(\omega_3/3)$ expressed thanks to (35). After some calculation, we get:

$$\begin{aligned}\wp_{1,4}(\omega) = \frac{1}{\omega^2} &+ \big[6\wp_{1,3}(\omega_3/3)^2 - 9g_{2,1,3}/20\big]\omega^2 \\ &+ \big[10\wp_{1,3}(\omega_3/3)^3 - 3\wp_{1,3}(\omega_3/3)g_{2,1,3}/2 - 27g_{3,1,3}/28\big]\omega^4 + O(\omega^6).\end{aligned} \tag{37}$$

Lemma 21 follows then as we identify the expansions (36) and (37). □

In the next lemma, we compute $\wp_{1,3}(\omega_3/3)$.

Lemma 22. *We have $\wp_{1,3}(\omega_3/3) = K$, where K is found as the only real positive solution to Equation (10).*

Proof. It is stated in Exercise 7 on page 182 in [9] that the quantity $\wp_{1,3}(\omega_3/3)$ is the only real positive solution to $K^4 - g_{2,1,3}K^2/2 - g_{3,1,3}K - g_{2,1,3}^2/48 = 0$. Replacing $g_{2,1,3}$ and $g_{3,1,3}$ with their expression obtained in Lemma 21, we conclude that $\wp_{1,3}(\omega_3/3)$ is a root of

$$K^4 - G_2K^2/2 - G_3K - G_2^2/48. \tag{38}$$

Let us now show that for any $z \in]0, 1/4[$, the polynomial (38) has a unique real positive root. Since $\wp_{1,3}(\omega_3/3) > 0$—indeed, $\wp_{1,3}$ is positive on $[0, \omega_3]$, see Part 6.11 on pages 166–167 in [9]—$\wp_{1,3}(\omega_3/3)$ shall be characterized as the only real positive solution to (10).

According to Lemma 23, it is now enough to verify that $G_2 \neq 0$ and that $G_2^3 - 27G_3^2 > 0$. The first fact is actually an immediate consequence of (9), while the second comes from the identity $G_2^3 - 27G_3^2 = (4^{14}/3^6)z^2(z-1/4)^4(z+1/4)^4$, see also (9). □

Lemma 23. *For any real numbers G_2, G_3 such that $G_2 \neq 0$ and $G_2^3 - 27G_3^2 > 0$, the polynomial (38) has a real negative root, a real positive root and two non-real complex conjugate roots.*

Proof. If some real numbers G_2, G_3 such that $G_2 \neq 0$ and $G_2^3 - 27G_3^2 \neq 0$ are given, then there exists a lattice $\hat{\omega}\mathbb{Z} + \breve{\omega}\mathbb{Z}$ and a Weierstrass elliptic function w.r.t. this lattice, say $\breve{\wp}$, having for invariants G_2, G_3, see Corollary 6.5.8 on page 287 in [6]. By using once again Exercise 7 on page 182 in [9], we then come to the conclusion that $\breve{\wp}(\hat{\omega}/3), \breve{\wp}(\breve{\omega}/3)$, $\breve{\wp}([\hat{\omega} - \breve{\omega}]/3), \breve{\wp}([\hat{\omega} + \breve{\omega}]/3)$ are the four roots of the polynomial (38).

Now we prove that the inequality $G_2^3 - 27G_3^2 > 0$ yields that one of $\breve{\wp}(\hat{\omega}/3), \breve{\wp}(\breve{\omega}/3)$ is negative while the other is positive, and that $\breve{\wp}([\hat{\omega} - \breve{\omega}]/3), \breve{\wp}([\hat{\omega} + \breve{\omega}]/3)$ are complex conjugate to one another.

If $G_2^3 - 27G_3^2 > 0$, then the periods $\hat{\omega}, \breve{\omega}$ of $\breve{\wp}$ can be chosen such that $\hat{\omega} > 0$ and $\breve{\omega}/\imath > 0$. Indeed, with pages 110–111 (particularly Theorem 3.6.12) of [6], we conclude that $\hat{\omega}, \breve{\omega}$ are either real and purely imaginary (if the polynomial $4x^3 - G_2x - G_3$ has three real roots) or complex conjugate to one another (if $4x^3 - G_2x - G_3$ has only one real root); in addition, it is well known that $4x^3 - G_2x - G_3$ has three real roots if and only if $G_2^3 - 27G_3^2 > 0$ and only one real root if and only if $G_2^3 - 27G_3^2 < 0$. Moreover, on the parallelogram $[0, \hat{\omega}] \times [-\breve{\omega}/(2\imath), \breve{\omega}/(2\imath)]$, $\breve{\wp}$ takes real values on the segments $[0, \hat{\omega}]$, $[-\breve{\omega}/2, \breve{\omega}/2]$, $[0, \hat{\omega}] \pm \breve{\omega}/2$, $[-\breve{\omega}/2, \breve{\omega}/2] + \hat{\omega}/2$, and $[-\breve{\omega}/2, \breve{\omega}/2] + \hat{\omega}$ and on those segments only, see Part 3.16 on pages 109–115 in [6].

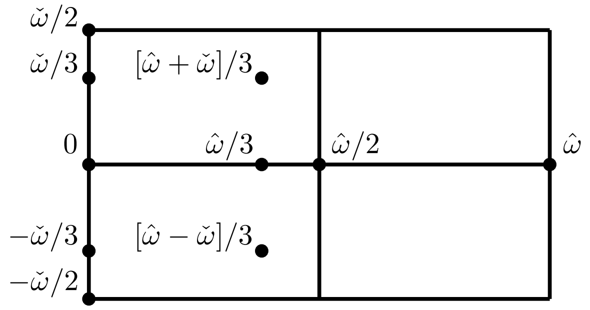

Figure 4: The parallelogram $[0, \hat{\omega}] \times [-\breve{\omega}/(2\imath), \breve{\omega}/(2\imath)]$

As a first consequence, $\breve{\wp}(\hat{\omega}/3)$ and $\breve{\wp}(\breve{\omega}/3)$ are real. Furthermore, on the segments $[0, \hat{\omega}]$ and $[-\breve{\omega}/2, \breve{\omega}/2]$, the values of $\breve{\wp}$ are positive and negative respectively, see Part 6.11 on pages 166–167 in [9]. This is why $\breve{\wp}(\hat{\omega}/3) > 0$ and $\breve{\wp}(\breve{\omega}/3) < 0$.

As a second consequence, $\breve{\wp}([\hat{\omega} - \breve{\omega}]/3)$ and $\breve{\wp}([\hat{\omega} + \breve{\omega}]/3)$ are non-real. Since $[\hat{\omega} - \breve{\omega}]/3$ and $[\hat{\omega} + \breve{\omega}]/3$ are complex conjugate to one another, so are $\breve{\wp}([\hat{\omega} - \breve{\omega}]/3)$ and $\breve{\wp}([\hat{\omega} + \breve{\omega}]/3)$, see also Part 6.11 on pages 166–167 in [9]. Lemma 23 is proved. $\square$

Remark 24. Lemma 23 is also valid for any real numbers G_2, G_3 such that $G_2 \neq 0$ and $G_2^3 - 27G_3^2 < 0$. On the other hand, if $G_2^3 - 27G_3^2 = 0$, then the polynomial (38) has only real roots.

Substituting in (34) equality (35) for $\wp_{1,3}'(\omega)$ and $\wp_{1,3}'(\omega_3/3)$, we express $\wp_{1,4}(\omega)$ in terms of $\wp_{1,3}(\omega)$, $g_{2,1,3}$, $g_{3,1,3}$ and $\wp_{1,3}(\omega_3/3)$. Applying then Lemmas 21 and 22, we get:

$$\begin{aligned}\wp_{1,3}(\omega)^3 - \wp_{1,3}(\omega)^2\big[\wp_{1,4}(\omega)+2K\big] \;+\; &\wp_{1,3}(\omega)\big[2K\wp_{1,4}(\omega)+K^2/3+G_2/2\big]\\ &-\;\big[K^2\wp_{1,4}(\omega)+19G_2K/18+G_3-46K^3/27\big]=0.\end{aligned}$$

In particular, evaluating this identity at $\omega = \wp^{-1}(f(t)) - [\omega_1+\omega_2]/2$, using (32) and taking into account (26), we obtain Equation (12) for the CGF w defined in (25).

If F, defined in (11), is infinite at some point t_0, then Equation (12) yields $[w(t_0) - K]^2 = 0$. Thus K is a double root of (12) at t_0. In addition, the double root K being non-zero by its definition via (10) and the product $K^2F(t_0) + 19G_2K/18 + G_3 - 46K^3/27$ of all roots of the polynomial (12) being infinite, the third root of (12) must be infinite.

Since F is infinite at x_2, see (11), and since w has a pole at x_2, see Subsection 5.2, w can thus be characterized as the unique solution to (12) with a pole at x_2—the two other solutions are finite at x_2 and equal to K. The part of Theorem 7 dealing with a CGF for the set $\mathscr{G}X([y_1, y_2])$ is concluded.

Now we prove the corresponding fact for $\mathscr{G}Y([x_1, x_2])$. Since w is a solution to (12) and since an easy calculation gives $F(X_0(t)) = \widetilde{F}(t)$, we conclude that $\widetilde{w}$ satisfies the equation obtained from (12) by replacing $F(t)$ by $\widetilde{F}(t)$. Furthermore, both $\widetilde{F}$ and $\widetilde{w}$ have a pole at x_3: for $\widetilde{F}$, this is a consequence of (11) and for $\widetilde{w}$, this follows from Lemma 16. Using the same arguments as above for w, we then derive that $\widetilde{w}$ can be characterized as the only function having a pole at x_3 and solution to the equation obtained from (12) by replacing $F(t)$ by $\widetilde{F}(t)$. □

6 Holomorphic continuation of $zQ(x, 0, z)$, $z(y+1)Q(0, y, z)$

In this part, we shall prove Theorem 3. In other words, we shall show that $zQ(x, 0, z)$ and $z(y + 1)Q(0, y, z)$ can be holomorphically continued from their unit disc up to $\mathbb{C} \setminus [x_3, x_4]$ and $\mathbb{C} \setminus [y_3, y_4]$ respectively.

Proof of Theorem 3. First of all, we lift the functions $Q(x, 0, z)$ and $Q(0, y, z)$ up to $\mathbb{C}/\Omega$ by setting $q_x(\omega) = Q(x(\omega), 0, z)$ and $q_y(\omega) = Q(0, y(\omega), z)$. We recall that $x(\omega)$ and $y(\omega)$, the coordinates of the uniformization, are defined in (19). The functions q_x and q_y are a priori well defined on $x^{-1}(\{|x| \leq 1\})$ and $y^{-1}(\{|y| \leq 1\})$ respectively. Then, we use the following result, that we shall prove in a few lines.

Theorem 25. *$q_x(\omega)$ and $(y(\omega) + 1)q_y(\omega)$, initially well defined on $x^{-1}(\{|x| \leq 1\})$ and $y^{-1}(\{|y| \leq 1\})$ respectively, can be holomorphically continued up to the whole parallelogram $\mathbb{C}/\Omega$ cut along $[0, \omega_1[$ and $[0, \omega_1[+\omega_3/2$ respectively. Moreover, these continuations satisfy*

$$q_x(\omega) = q_x(\psi(\omega)),\ \forall\omega \in \mathbb{C}/\Omega \setminus [0, \omega_1[, \qquad q_y(\omega) = q_y(\phi(\omega)),\ \forall\omega \in \mathbb{C}/\Omega \setminus ([0, \omega_1[+\omega_3/2), \tag{39}$$

and

$$zq_x(\omega) + z(y(\omega) + 1)q_y(\omega) - zQ(0, 0, z) - x(\omega)y(\omega) = 0,\ \forall\omega \in]3\omega_2/8, \omega_2[\times[0, \omega_1/\imath[. \tag{40}$$

Finally, we set $Q(x,0,z)=q_x(\omega)$ if $x(\omega)=x$ as well as $Q(0,y,z)=q_y(\omega)$ if $y(\omega)=y$. Thanks to (39) and Proposition 13, these equalities define not ambiguously $Q(x,0,z)$ and $Q(0,y,z)$ on $\mathbb{C}\setminus[x_3,x_4]$ and $\mathbb{C}\setminus[y_3,y_4]$ respectively, as holomorphic functions. Furthermore, the statements (4) and (5) are immediate consequences of (40). $\square$

The proof of Theorem 3 is now reduced to that of Theorem 25. In order to carry out this proof, we need to find the location of the cycles $x^{-1}(\{|x|=1\})$ and $y^{-1}(\{|y|=1\})$ on $\mathbb{C}/\Omega$. This is the subject of the following result, illustrated on Figure 5 below.

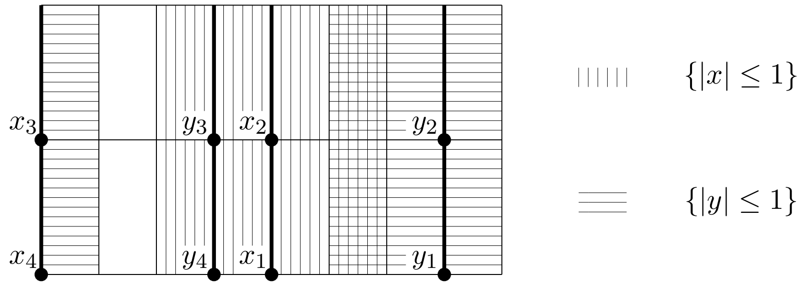

Figure 5: Location of the important cycles on the surface $[0,\omega_2[\times[0,\omega_1/\imath[$

Proposition 26. *We have* $x^{-1}(\{|x|=1\})=([0,\omega_1[+\omega_2/4)\cup([0,\omega_1[+3\omega_2/4)$ *as well as* $y^{-1}(\{|y|=1\})=([0,\omega_1[+\omega_2/8)\cup([0,\omega_1[+5\omega_2/8)$.

Proof. The details are of course essentially the same for x and y, so that we are going to prove only the assertion dealing with x.

First of all, we note that because of the equality $x\circ\psi=x$, it is sufficient to prove that $x^{-1}(\{|x|=1\})\cap([0,\omega_2/2[\times[0,\omega_1/\imath[)=[0,\omega_1[+\omega_2/4$—the advantage of this is that $\wp$, and therefore also x, are one-to-one in the half-parallelogram $[0,\omega_2/2[\times[0,\omega_1/\imath[$.

The proof is then composed of three steps.

Step 1. We prove that $x(\omega_2/4+\omega_1/2)=1$. For this, we recall that $\wp(\omega_2/4)=(1+4z^2)/3$, $\wp'(\omega_2/4)=-8z^2$, $\wp(\omega_1/2)=f(x_3)$ with f defined in (22) and $\wp'(\omega_1/2)=0$, see Lemma 19 and its proof. With addition formula (30), we then immediately obtain the explicit value of $\wp(\omega_2/4+\omega_1/2)$. Finally, by using Equation (19) and after a simple calculation, we get $x(\omega_2/4+\omega_1/2)=1$.

Step 2. We show that $x^{-1}(\{|x|=1\})\cap([0,\omega_2/2[\times[0,\omega_1/\imath[)\subset[0,\omega_1[+\omega_2/4$. For this, let $\theta\in[0,2\pi[$ and consider the equation $x(\omega)=\exp(\imath\theta)$. Thanks to (19) and (22), we obtain $\wp(\omega)=f(\exp(\imath\theta))$ and thus $\omega=\wp^{-1}(f(\exp(\imath\theta)))$. We can then use the explicit expression of the inverse function of $\wp$ on $[0,\omega_2/2[\times[0,\omega_1/\imath[$, see e.g. Part 6.12 on pages 167–172 in [9], and we get

$$\omega=x^{-1}(1)+\int_{f(1)}^{f(\exp(\imath\theta))}\frac{\mathrm{d}t}{[4t^3-g_2t-g_3]^{1/2}}=\omega_2/4+\omega_1/2+\frac{1}{2}\int_{\exp(\imath\theta)}^{1}\frac{\mathrm{d}x}{[d(x)]^{1/2}},\qquad(41)$$

where d is defined in Section 2 and g_2, g_3 in Lemma 12. Note that the second equality above comes from the first step and from exactly the same calculations as in Part 3.3 of [4]. Now we notice that $d(x) = x^4 d(1/x)$. In particular, the change of variable $x \mapsto 1/x$ in the integral $\int_{\exp(\imath\theta)}^{1} \mathrm{d}x/[d(x)]^{1/2}$ yields $\int_{\exp(\imath\theta)}^{1} \mathrm{d}x/[d(x)]^{1/2} = -\int_{\exp(-\imath\theta)}^{1} \mathrm{d}x/[d(x)]^{1/2}$. As a consequence, this integral belongs to $\imath\mathbb{R}$. In conclusion, with (41), we have shown that $x^{-1}(\{|x| = 1\}) \cap ([0, \omega_2/2[\times [0, \omega_1/\imath[) \subset [0, \omega_1[+\omega_2/4$.

Step 3. We prove that the inclusion above has to be an equality. Indeed, if it was not the case, the curve $x^{-1}(\{|x| = 1\}) \cap ([0, \omega_2/2[\times[0, \omega_1/\imath[)$ would be *not* closed, which is a manifest contradiction with the facts that $\{|x| = 1\}$ is closed and that x is meromorphic as well as one-to-one in the half-parallelogram $[0, \omega_2/2[\times[0, \omega_1/\imath[$. □

Proof of Theorem 25. This proof is composed of two steps: at first, we shall define the continuations of q_x and q_y on the parallelogram $\mathbb{C}/\Omega$ cut along $[0, \omega_1[$ and $[0, \omega_1[+\omega_3/2$ respectively; then, we shall verify that these continuations satisfy the conclusions of Theorem 25.

Step 1. We define the continuations of q_x and q_y.

(i) We define $q_x(\omega)$ on $x^{-1}(\{|x| \leq 1\})$ by $Q(x(\omega), 0, z)$ and $q_y(\omega)$ on $y^{-1}(\{|y| \leq 1\})$ by $Q(0, y(\omega), z)$. Note that as a consequence of Proposition 26, we have $x^{-1}(\{|x| \leq 1\}) = [\omega_2/4, 3\omega_2/4] \times [0, \omega_1/\imath[$ and $y^{-1}(\{|y| \leq 1\}) = [5\omega_2/8, 9\omega_2/8] \times [0, \omega_1/\imath[$.

(ii) Motivated by (1), on $[3\omega_2/4, \omega_2[\times[0, \omega_1/\imath[\subset y^{-1}(\{|y| \leq 1\})$, we set $q_x(\omega) = -(y(\omega)+1)q_y(\omega) + Q(0, 0, z) + x(\omega)y(\omega)/z$ and on $]3\omega_2/8, 5\omega_2/8] \times [0, \omega_1/\imath[\subset x^{-1}(\{|x| \leq 1\})$, we set $(y(\omega) + 1)q_y(\omega) = -q_x(\omega) + Q(0, 0, z) + x(\omega)y(\omega)/z$.

(iii) On $]0, \omega_2/4] \times [0, \omega_1/\imath[$, we define $q_x(\omega)$ by $q_x(\phi(\omega))$. Note that with Equation (24), we have $\phi(]0, \omega_2/4] \times [0, \omega_1/\imath[) = [3\omega_2/4, \omega_2[\times[0, \omega_1/\imath[$. On $[\omega_2/8, 3\omega_2/8[\times[0, \omega_1/\imath[$, we define $q_y(\omega)$ by $q_y(\psi(\omega))$. By using (24), we have $\psi([\omega_2/8, 3\omega_2/8[\times[0, \omega_1/\imath[) =]3\omega_2/8, 5\omega_2/8] \times [0, \omega_1/\imath[$.

The functions q_x and q_y are now well defined on the whole parallelogram $\mathbb{C}/\Omega$ cut along $[0, \omega_1[$ and $[0, \omega_1[+\omega_3/2$ respectively.

Step 2. We prove that the continuations of q_x and q_y defined in the first step satisfy the different assertions of Theorem 25.

Let us verify Equation (39) for q_x. By using (i) as well as the equality $x \circ \psi = x$, (39) is manifestly satisfied on $[\omega_2/4, 3\omega_2/4] \times [0, \omega_1/\imath[= \psi([\omega_2/4, 3\omega_2/4] \times [0, \omega_1/\imath[)$. Moreover, with (iii), (39) is satisfied for q_x on $]0, \omega_2/4] \times [0, \omega_1/\imath[$. Since $\psi^2 = \mathrm{id}$, (39) is also true for q_x on $[3\omega_2/4, \omega_2[\times[0, \omega_1/\imath[$ and thus finally on the whole $\mathbb{C}/\Omega \setminus [0, \omega_1[$. Likewise, we easily verify that (39) is valid for q_y on $\mathbb{C}/\Omega \setminus ([0, \omega_1[+3\omega_2/8)$. Equation (40) is immediately true, by construction of the continuations.

It remains to prove that the continuations of q_x and $(y+1)q_y$ are holomorphic on $\mathbb{C}/\Omega$ cut along $[0, \omega_1[$ and $[0, \omega_1[+3\omega_2/8$ respectively.

We first show that q_x is meromorphic on its respective cut parallelogram. The following cycles are a priori problematic for q_x: $[0, \omega_1[$, $[0, \omega_1[+\omega_2/4$ and $[0, \omega_1[+3\omega_2/4$. In an open neighborhood of $[0,\omega_1[+3\omega_2/4$, we have $q_x(\omega)=-(y(\omega)+1)q_y(\omega)+Q(0,0,z)+x(\omega)y(\omega)/z$, so that q_x is in fact meromorphic in the neighborhood of the cycle $[0,\omega_1[+3\omega_2/4$. Since Equation (39) holds, q_x is also meromorphic near $[0,\omega_1[+\omega_2/4=\psi([0,\omega_1[+3\omega_2/4)$. Thus $[0,\omega_1[$ remains the only singular cycle for q_x. Furthermore, q_x is clearly holomorphic on $]\omega_2/4,3\omega_2/4]\times[0,\omega_1/\imath[$, since it is defined there through the power series $Q(x,0,z)$. On $]5\omega_2/8,\omega_2[\times[0,\omega_1/\imath[$, we have $q_x(\omega)=-(y(\omega)+1)q_y(\omega)+Q(0,0,z)+x(\omega)y(\omega)/z$ and the first two terms in the right-hand side of this equality are holomorphic on this domain. With Lemma 28 below, the product $x(\omega)y(\omega)$ may have a pole on this domain only at $7\omega_2/8$: in fact, x has a pole of order one at this point but y has there a zero of order two, so that the product $x(\omega)y(\omega)$ is holomorphic near $7\omega_2/8$. On $]0,3\omega_2/8[\times[0,\omega_1/\imath[$, we have $q_x=q_x\circ\psi$, so that q_x is holomorphic on this domain, since it is on $\psi(]0,3\omega_2/8[\times[0,\omega_1/\imath[)=]5\omega_2/8,\omega_2[\times[0,\omega_1/\imath[$.

A similar reasoning yields that $(y+1)q_y$ is holomorphic on $\mathbb{C}/\Omega\setminus([0,\omega_1[+3\omega_2/8)$. □

Corollary 27. *The function q_y is holomorphic on $\mathbb{C}/\Omega\setminus([0,\omega_1[+3\omega_2/8)$.*

Proof. From Theorem 25, we know that $(y+1)q_y$ is holomorphic on $\mathbb{C}/\Omega\setminus([0,\omega_1[+3\omega_2/8)$, so that we directly derive that q_y is holomorphic on the same domain, except eventually at the points where $y+1=0$, i.e. at $\omega_2/8$ and $5\omega_2/8$, see Lemma 28. However, the generating function $Q(0,y,z)$ is bounded at $y=-1$, see Section 2, so that $q_y(\omega)=Q(0,y(\omega),z)$, being meromorphic and bounded near $\omega_2/8$ and $5\omega_2/8$, is actually holomorphic at both these points. □

The following lemma, which has been used in the proof of Theorem 25, easily follows from Lemma 1 and from the fact that on the parallelogram $[0,\omega_2[\times[0,\omega_1/\imath[$, the Weierstrass elliptic function $\wp$ takes each value of $\mathbb{C}\cup\{\infty\}$ twice.

Lemma 28. *The only poles of x (of order one) are at $\omega_2/8,7\omega_2/8$ and its only zeros (of order one) are at $3\omega_2/8,5\omega_2/8$. The only pole of y (of order two) is at $3\omega_2/8$ and its only zero (of order two) is at $7\omega_2/8$. The only zeros of $y+1$ are at $\omega_2/8,5\omega_2/8$.*

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